Abstract Algebra

General properties of Groups

Theorem 3:- If a, b, c are three elements of a group $(G, *)$ then

$$(a*c) = b*c \Rightarrow a = b \quad \text{(Right cancellation law)}$$

$$c*a = c*b \Rightarrow a = b \quad \text{(Left cancellation law)}$$

Proof:- (i)
$$a*c = b*c \Rightarrow (a*c)*c^{-1} = (b*c)*c^{-1} \quad (\because c^{-1} \in G)$$
$$\Rightarrow a*(c*c^{-1}) = b*(c*c^{-1}) \quad \text{(By associativity)}$$
$$\Rightarrow a*e = b*e \Rightarrow a = b$$

$$c*a = c*b \Rightarrow c^{-1}*c*a = c^{-1}*(c*b)$$
$$\Rightarrow (c^{-1}*c)*a = (c^{-1}*c)*b$$
$$\Rightarrow e*a = e*b \Rightarrow a = b$$

Theorem 4:- In a group G, the equation $a*x = b$ and $y*a = b$ where $a, b \in G$ have solutions in G.

Proof:-
$$a*x = b$$
$$\Rightarrow a^{-1}*(a*x) = a^{-1}*b \quad (\because a^{-1} \in G)$$
$$\Rightarrow (a^{-1}*a)*x = a^{-1}*b \quad \text{(by associativity)}$$
$$\Rightarrow e*x = a^{-1}*b$$
$$\Rightarrow x = a^{-1}*b \in G \quad (\because a^{-1}, b \in G \Rightarrow a^{-1}*b \in G)$$

Therefore, the equation $a*x = b$ has a solution $x = a^{-1}*b$ in G. Similarly, the equation $y*a = b$ has a solution $y = b*a^{-1}$ in G.

Uniqueness :-

Let, if possible x_1 and x_2 be any two solutions of the equation $a * x = b$ so that

$$a * x_1 = b \quad \text{and} \quad a * x_2 = b$$

$$\Rightarrow a * x_1 = a * x_1 = a * x_2 \Rightarrow x_1 = x_2$$

(By left cancellation law)

Therefore, the solution of the equation $a * x = b$ is unique. Similarly, it can be proved that the equation $y * a = b$ has a unique solution.

Hence, the given equation have unique solutions in G.

Remark

(1) with the help of the above theorem, we can define the group alternatively as follows:

"A set G with a binary composition $*$ is a group iff"

(i) the composition $*$ is associative

(ii) the equations $ax = b$ and $ya = b$ have unique solutions in G.