Abstract Algebra

GROUPS

Let G be a non-empty set and $*$ be a binary operation defined on it, then the structure $(G, *)$ is said to be a group if the following axioms are satisfied:

(i) Closure property:

$$a * b \in G, \forall a, b \in G$$

(ii) Associativity

The operation $*$ is associative on G, i.e.

$$a * (b * c) = (a * b) * c; \forall a, b, c \in G$$

(iii) Existence of identity

There exists an element $e \in G$ such that

$$a * e = e * a; \forall a \in G.$$

e is called identity of $*$ in G.

(iv) Existence of inverse

For each element $a \in G$, there exist an element $b \in G$ such that $a * b = b * a = e$. The element b is called the inverse of element a with respect to $*$ and we write $b = a^{-1}$

ABELIAN OR COMMUTATIVE GROUP

A group $(G, *)$ is said to be abelian or Commutative if $a * b = b * a$, $\forall\ a, b \in G$. The group which are not abelian are called non-abelian or non-Commutative.

FINITE AND INFINITE GROUPS

If a group contains a finite number of elements, it is called a finite group. If the number of elements in a group is infinite, it is called a infinite group.

ORDER OF A GROUP

The number of elements in a finite group is called the order of the group. It is denoted by $o(G)$.

An infinite group is called a group of infinite order.

EXAMPLES OF GROUPS

(i) The set Z of integers is an infinite abelian group with respect to the operation of addition but Z is not a group with respect to the multiplication.

(ii) Let $G = \{1\}$, then G is an abelian group of order 1 with respect to multiplication

(iii) Let $G = \{0\}$, then G is an abelian group of order 1 with respect to addition.

(iv) Let $G = \{1, -1\}$, then G is an abelian group of order 2 with respect to multiplication.

Remarks

① when we say $*$ is a binary operation defined on a non-empty set G, it implies that G is closed for the binary operation $*$, i.e.

$$a \in G,\ b \in G \Rightarrow a*b \in G \ \forall\ a, b \in G$$

② A group is not simply a set, but it is an algebraic structure.

③ Because of the associativity, the parenthesis can be dropped in products of more than two elements of a group and instead of writing $a*(b*c)$ or $(a*b)*c$ we may simply write $a*b*c$. The associative law can be extended to any finite number of elements.

Remarks.

① An abelian group under addition is sometimes called a 'module'.

② The Commutative group is also known as Abelian group after the name of famous mathematician Abel.

③ The smallest group for a given composition is the set $\{e\}$, containing identity elements.

④ A group consisting of the identity element only, is called a trivial group; others are called non-trivial groups.