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Adjoint Operator:-

Let $V(F)$ be an inner product space and $T: V \to V$ be linear operator. Let $\exists$ (there exists) a unique operator T^* on V such that

$$\langle T(u), v \rangle = \langle u, T^*(v) \rangle \quad \text{①} \quad \forall\, u, v \in V$$

Then T^* is called adjoint of T.

To prove that T^* is linear

Let $u, v, w \in V$ and $\alpha, \beta \in F$ ($F \to$ field)

$$\langle u, T^*(\alpha v + \beta w) \rangle = \langle T(u), (\alpha v + \beta w) \rangle \quad \text{by def ①}$$

$$= \langle T(u), \alpha v \rangle + \langle T(u), \beta w \rangle$$

by inner product property

$$= \bar{\alpha} \langle T(u), v \rangle + \bar{\beta} \langle T(u), w \rangle$$

$$= \bar{\alpha} \langle u, T^*(w) \rangle + \bar{\beta} \langle u, T^*(w) \rangle$$

$$= \langle u, \alpha T^*(u) \rangle + \langle u, \beta T^*(w) \rangle$$

Thus $\langle u, T^*(\alpha v + \beta w) \rangle = \langle u, \alpha T^*(u) \rangle + \langle u, \beta T^*(w) \rangle$

$$\langle u, T^*(\alpha u + \beta w) \rangle = \langle u, \alpha T^*(u) + \beta T^*(w) \rangle$$

This is True $\forall\, u \in V$

Hence the $T^*(\alpha v + \beta w)$ gives

$$T^*(\alpha v + \beta w) = \alpha T^*(v) + \beta T^*(w)$$

This prove that T^* is linear

* To show that T^* is unique. If possible let S be another linear operator on V such that

$$\langle T(u), v\rangle = \langle u, S(v)\rangle$$

$$\langle u, T^*(v)\rangle = \langle u, S(v)\rangle \quad \forall\ u, v \in V$$

This $\Rightarrow$ $T^*(v) = S(v)$ $\Rightarrow$ $T^* = S$

$\Rightarrow$ T^* is unique.

<u>Definition of Self-Adjoint Operator</u>:

A linear operator T on inner product space $V(F)$ is called self-adjoint operator iff $T = T^*$

i.e. iff $\langle T(u), v\rangle = \langle u, T(v)\rangle \quad \forall\ u, v \in V$

A self-adjoint operator is called symmetric or Hermitian according as the space is Euclidean (i.e. $F = R$, field is real) or Unitary ($F = C$ e.g. field is complex)

<u>Exp.</u> Let A is $m\times n$ matrix over field F i.e. $A \in M_{m\times n}(F)$. The conjugate transpose or adjoint of A to the $n\times m$ matrix A^* such that $(A^*)_{ij} = \bar{A}_{ji} \quad \forall\ i, j$

or $A^*_{ij} = (\bar{A}_{ij})^T$

$$A_{2\times 2} = \begin{bmatrix} i & 1+2i \\ 2 & 3+4i \end{bmatrix} \Rightarrow \bar{A}_{2\times 2} = \begin{bmatrix} -i & 1-2i \\ 2 & 3-4i \end{bmatrix} \Rightarrow$$

$$[\bar{A}_{2\times 2}]^T = \begin{bmatrix} -i & 2 \\ 1-2i & 3-4i \end{bmatrix} \Rightarrow A^*$$ This is adjoint of $A_{2\times 2}$ matrix.

Hermitian Matrix – A complex square matrix $A = [a_{ij}]$ is said to be Hermitian if $A = A^*$ i.e. $[a_{ij}] = [\bar{a}_{ji}]$

Exp.

$$A = \begin{bmatrix} 2 & 2-3i & 3+4i \\ 2+3i & 3 & 4+5i \\ 3-4i & 4-5i & 4 \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 2 & 2+3i & 3-4i \\ 2+3i & 3 & 4-5i \\ 3+4i & 4+5i & 4 \end{bmatrix}$$

$$[\bar{A}]^T = \begin{bmatrix} 2 & 2-3i & 3+4i \\ 2+3i & 3 & 4+5i \\ 3-4i & 4-5i & 4 \end{bmatrix}$$

$$A = [\bar{A}]^T = A^*$$

Hence A^* is self-adjoint of A matrix.

If $A^* = -A$ i.e. $[a_{ij}] = -[\bar{a}_{ji}]$ is said to Skew-Hermetian matrix.