Differential Calculus :-

In this, study of fundamental ideas of the Integration & Differential Calculus.

→ The fundamental limiting processes of calculus are integration and differentiation. $I = \int_a^b f(x)\,dx$

i.e. $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

Differentiation (the concept of rate of change)

It is the concept needed for describing the notions of tangents to curves - and of velocity of moving particles, or more generally, the concept of rate of change. Thus, differentiation is the derivative of the function $y = f(x)$ with respect to x, denoted as $f'(x)$ or $\frac{dy}{dx}$.

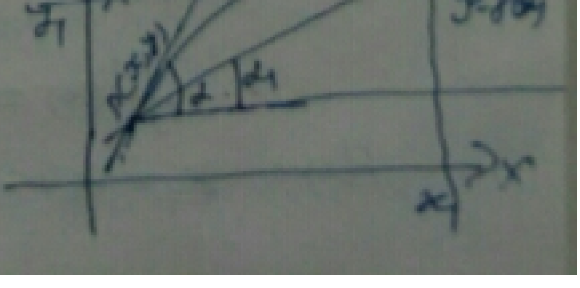
It is define as the slope of the tangent line to the curve $y = f(x)$ at the point $P(x, y)$

This slope is obtained by a Limit as: $\left(\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}\right)$

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Geometrical Definition :-

We define the tangent to the given curve $y = f(x)$ at one of its points by means of the following geometrical limiting process.



Let $P_1(x+\Delta x, y+\Delta y)$ near $P(x, y)$ on the curve.

PP_1 is the secant of the curve.

→ If now the point P_1 moves along the curve towards the point P, then the secant is expected to approach a limiting position which is independent of the side from which P_1 tends to P.

Hence, this limiting position of the secant is the tangent; the statement such that a limiting position of the secant exists is a equivalent to the assumption that curve has a definite tangent or a definite direction at the point P.

Since our curve is represented by means of a function $y=f(x)$, we must formulate the geometric limiting process analytically, with reference to $f(x)$. This analytical limit process is called differentiation of $f(x)$.

Let α_1 be the angle which the secant (PP_1) with x-axis and α angle which the tangent (T) forms with the x-axis

Then $\lim\limits_{P_1 \to P} \alpha_1 = \alpha$

$$\tan\alpha_1 = \frac{y_1 - y}{x_1 - x} = \frac{f(x_1) - f(x)}{x_1 - x} = \frac{\Delta y}{\Delta x}$$

$$\lim_{x_1 \to x} \frac{f(x_1) - f(x)}{x_1 - x} = \lim_{x_1 \to x} \tan\alpha_1 = \tan\alpha$$

$$f'(x) = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Here $x_1 = x + \Delta x$ & $y_1 = y + \Delta x$

Successive Differentiation

The process of differentiating a function again and again is called successive differentiation.

or- A function is differentiating n-times called successive differentiation of nth-derivatives.

If $y = f(x)$, then its successive derivatives are denoted by- $\underbrace{\frac{dy}{dx}}_{\text{first der.}}, \underbrace{\frac{d^2y}{dx^2}}_{\text{second der.}} \cdots \underbrace{\frac{d^ny}{dx^n}}_{\text{nth-deriv.}}$.

$\boxed{\frac{dy}{dx} \rightarrow \text{steepness of a curve at point } (x, y)}$

= slope of a tangent to the curve at the point (x, y), where y is a function of x i.e. $y = f(x)$

$\frac{dy}{dx}$ or $f'(x)$ or y_1 is called the first derivative of y w.r.t x.

Exp. To find the nth derivatives of x^m.

Sol. Given that
$$y = x^m \quad \text{——} (1)$$

Differentiating Eqn (1) both sides w.r.t x again and again, we get

$$\frac{dy}{dx} = y_1 = m x^{m-1}$$

$$\frac{d^2y}{dx^2} = y_2 = m(m-1) x^{m-2}$$

$$\frac{d^3y}{dx^3} = y_3 = m(m-1)(m-2) x^{m-3}$$

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$$\frac{d^ny}{dx^n} = y_n = m(m-1)(m-2) \cdots (m-(n-1)) x^{m-n} \quad \text{——} (2)$$

Again, differentiating Eqn (2), we get

$$\frac{d^{n+1}y}{dx^{n+1}} = y_{n+1} = m(m-1)(m-2)\cdots(m-n+1)(m-n)x^{m-(n+1)}$$

Hence, y_{n+1} is of the same form y_n, given by equation (2). This $\Rightarrow$ If Eqn (2) is true for n value then Eqn (3) is true for next higher value of n also. It is true for n=1, 2, 3 and so on.

From Eqn (2) we can write y_n as follows:-

$$y_n = \frac{m(m-1)(m-2)\cdots\cdots(m-(n-1))(m-n)(m-(n+1))\cdots 3.2.1}{(m-n)(m-(n+1))\cdots 3.2.1} x^{m-n}$$

$$\boxed{y_n = \frac{m!}{(m-n)!}x^{m-n}} \quad \text{——— (3)}$$

Corollary if $y = x^n$, then $y_n = n!$

$\because \quad y = x^m \Rightarrow y_n = \frac{m!}{(m-n)!}x^{m-n}$

Put $m = n$ then

$$y = x^n \Rightarrow y_n = \frac{n!}{(n-n)!}x^{n-n} = \frac{n!}{1}\cdot 1 = \underline{\underline{\lfloor n}}$$

From (3) for $n=1$

$$y_1 = \frac{m!}{(m-1)!}x^{m-1} = \frac{m.\cancel{\lfloor m-1}}{\cancel{\lfloor m-1}}x^{m-1} = mx^{m-1}$$

for $n=2$

$$y_2 = \frac{\lfloor m}{\lfloor m-2}\cdot x^{m-2} = \frac{m(m-1)\cancel{\lfloor m-2}}{\cancel{\lfloor m-2}}x^{m-2}$$

$$= m(m-1)x^{m-2}$$

for $n=3$

$$y_3 = \frac{\lfloor m}{\lfloor m-3}\cdot x^{m-3} = \frac{m(m-1)(m-2)\cancel{\lfloor m-3}}{\cancel{\lfloor m-3}}x^{m-3}$$

$$y_3 = m(m-1)(m-2)x^{m-3}$$

and so on. - - - =