Exp. If $y = \frac{x}{x^2+16}$, find y_n

Solution Given that

$$y = \frac{x}{x^2+16} = \frac{x}{(x+4i)(x-4i)}$$

$$y = \frac{1}{2}\left[\frac{1}{x-4i} + \frac{1}{x+4i}\right]$$

$$y_n = \frac{1}{2}\left[\frac{(-1)^n \lfloor n}{(x-4i)^{n+1}} + \frac{(-1)^n \lfloor n}{(x+4i)^{n+1}}\right]$$

$$y_n = \frac{(-1)^n \cdot \lfloor n}{2}\left[\frac{1}{(x-4i)^{n+1}} + \frac{1}{(x+4i)^{n+1}}\right]$$

Now put $x = r\cos\theta$

$4 = r\sin\theta$

$r^2 = x^2+4 \Rightarrow r = \sqrt{x^2+4}$

$\tan\theta = \frac{4}{x} \Rightarrow \theta = \tan^{-1}\frac{4}{x}$

Then

$$y_n = \frac{(-1)^n \lfloor n}{2}\left[\frac{1}{(r\cos\theta - ir\sin\theta)^{n+1}} + \frac{1}{(r\cos\theta + ir\sin\theta)^{n+1}}\right]$$

$$= \frac{(-1)^n \lfloor n}{2}\left[\frac{1}{r^{n+1}(\cos\theta - i\sin\theta)^{n+1}} + \frac{1}{r^{n+1}(\cos\theta + i\sin\theta)^{n+1}}\right]$$

$$= \frac{(-1)^n \lfloor n}{2} \cdot \frac{1}{r^{n+1}}\left[\frac{1}{(e^{-i\theta})^{n+1}} + \frac{1}{(e^{i\theta})^{n+1}}\right]$$

$$y_n = \frac{(-1)^n \lfloor n}{2} \cdot \frac{1}{r^{n+1}}\left[e^{i(n+1)\theta} + e^{-i(n+1)\theta}\right]$$

$$= \frac{(-1)^n \lfloor n}{2} \frac{1}{r^{n+1}} \cdot 2 \cdot \cos(n+1)\theta$$

$$= \frac{(-1)^n \lfloor n}{r^{n+1}} \cdot \cos(n+1)\theta$$

But $r = \frac{4}{\sin\theta}$ $\because 4 = r\sin\theta$

$$y_n = \frac{(-1)^n \lfloor n}{(4)^{n+1}} \cdot \sin^{n+1}\theta \cdot \cos(n+1)\theta$$

where $\tan\theta = \frac{4}{x} \Rightarrow \theta = \tan^{-1}\frac{4}{x} = \cot^{-1}\frac{4}{x}$

__Ques.__ If $y = \frac{1}{x^2+16}$ to find y_n

__Sol.__ Similar solution of this as above Example

Hint. $y = \frac{1}{x^2+16} = \frac{1}{x^2 - 16i^2} = \frac{1}{8i}\left[\frac{1}{x-4i} - \frac{1}{x+4i}\right]$

$$y = \frac{1}{8i}\left[\frac{1}{x-4i} - \frac{1}{x+4i}\right]$$

$$\Rightarrow y_n = \frac{1}{8i}\left[\frac{(-1)^n \lfloor n}{(x-4i)^{n+1}} - \frac{(-1)^n \lfloor n}{(x+4i)^{n+1}}\right]$$

put $x = r\cos\theta$ & $4 = r\sin\theta$ and find y_n.