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For T.D.C. Part III

Paper-5

Gr- A

9. TAYLOR'S THEOREM (For the functions of two variables)

If $f(x, y)$ is a function which possesses continuous partial derivatives of order n in any domain of a point (a, b), and the domain is large enough to contain a point $(a + h, b + k)$ within it, then there exists a positive number, $0 < \theta < 1$, such that

$$f(a + h, b + k) = f(a, b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a, b)$$

$$+ \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a, b)$$

$$+ \dots + \frac{1}{(n-1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n-1} f(a, b) + R_n,$$

where $R_n = \dfrac{1}{n!}\left(h\dfrac{\partial}{\partial x} + k\dfrac{\partial}{\partial y}\right)^n f(a + \theta h, b + \theta k), 0 < \theta < 1.$

Let $x = a + th$, $y = b + tk$, where $0 \leqslant t \leqslant 1$ is a parameter, and

$f(x, y) = f(a + th, b + tk) = \phi(t)$ — (A)

If the function $f(x, y)$ is differentiable at a point (a, b) in its domain, then the partial derivatives f_x and f_y both exist and are finite.

Proof : Since the function is differentiable at (a, b), therefore

$$f(a+h, b+k) - f(a, b) = Ah + Bk + h\phi(h, k) + k\psi(h, k) \quad \text{...(1)}$$

where A and B are constants independent of h and k and $\phi(h, k)$, $\psi(h, k)$ are functions of h and k so that

$$\operatorname*{Lt}_{(h, k) \to (0, 0)} \phi(h, k) = 0 = \operatorname*{Lt}_{(h, k) \to (0, 0)} \psi(h, k).$$

Now, putting $k = 0$ and passing to the limit as $h \to 0$, we find from (1) that

$$\operatorname*{Lt}_{h \to 0} \frac{[f(a+h) - f(a, b)]}{h} = A$$

i.e.,

$$f_x(a, b) = A.$$

Similarly,

$$f_y(a, b) = B.$$

NOTE The converse is not true. For the converse let us see the following exam

Since the partial derivatives of $f(x, y)$ of order n are continuous in the domain under consideration, $\phi^n(t)$ is continuous in $[0, 1]$, and also

$$\phi'(t) = \frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f$$

$$\phi''(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f$$

$$\vdots$$

$$\phi^{(n)}(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f$$

therefore by Maclaurin's theorem

$$\phi(t) = \phi(0) + t\phi'(0) + \frac{t^2}{2!}\phi''(0) + \dots + \frac{t^{n-1}}{(n-1)!}\phi^{(n-1)}(0) + \frac{t^n}{n!}\phi^{(n)}(\theta t),$$

where $0 < \theta < 1$.

Now putting $t = 1$, we get

$$\phi(1) = \phi(0) + \phi'(0) + \frac{1}{2!}\phi''(0) + \dots + \frac{1}{(n-1)!}\phi^{(n-1)}(0) + \frac{1}{n!}\phi^{(n)}(\theta)$$

But $\phi(1) = f(a + h, b + k)$, and $\phi(0) = f(a, b)$ $= f(x,y)$ [from A

$$\phi'(0) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a, b)$$

$$\phi''(0) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a, b)$$

$$\vdots$$

$$\phi^{(n)}(\theta) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(a + \theta h, b + \theta k)$$

Putting all the above values in (B),

$$\therefore\ f(a + h, b + k) = f(a, b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a, b)$$

$$+ \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a, b) + \dots$$

$$+ \frac{1}{(n-1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n-1} f(a, b)$$

where $R_n = \dfrac{1}{n!}\left(h\dfrac{\partial}{\partial x} + k\dfrac{\partial}{\partial y}\right)^n f(a + \theta h, b + \theta k),\ 0 < \theta < 1.$

We now prove two theorems which lay down sufficient conditions for the equality of f_{xy} and f_{yx} at a point (a, b).

SCHWARTZ'S THEOREM on Differentiability

Theorem I. Let (a, b) be a point of the domain of a function f such that

(i) f_x exists in a certain neighbourhood of (a, b).

(ii) f_{xy} is continuous at (a, b).

Then $f_{yx}(a, b)$ exists and is equal to $f_{xy}(a, b)$.

Proof : The given conditions imply that there exists a certain nbd. of (a, b) at every point (x, y) of which $f_x(x, y)$, $f_y(x, y)$ and $f_{xy}(x, y)$ all exist.

Let $(a + h, b + k)$ be any point of the nbd.

We write $F(h, k) = f(a + h, b + k) - f(a + h, b) - f(a, b + k) + f(a, b)$

$$g(y) = f(a + h, y) - f(a, y)$$

so that $$F(h, k) = g(b + k) - g(b) \quad \text{... (1)}$$

Since f_y exists in a nbd. of (a, b), the function g of one variable y is derivable in $[b, b + k]$ and therefore by applying the mean-value theorem to the expression on the right of (1), we have

$$F(h, k) = kg'(b + \theta k), \text{ where } 0 < \theta < 1.$$

$$= k[f_y(a + h, b + \theta k) - f_y(a, b + \theta k)] \quad \text{... (2)}$$

Again, since f_{xy} exists in a nbd. of (a, b), therefore by applying the mean-value theorem to the right of (2), we get

$$F(h, k) = k[hf_{xy}(a + \theta' h, b + \theta k)] \text{ where } 0 < \theta' < 1$$

$$\Rightarrow \quad \frac{F(h, k)}{hk} = f_{xy}(a + \theta' h, b + \theta k)$$

$$\Rightarrow \quad \frac{1}{k}\left[\frac{f(a + h, b + k) - f(a, b + k)}{h} - \frac{f(a + h, b) - f(a, b)}{h}\right]$$

$$= f_{xy}(a + \theta' h, b + \theta k).$$

Since f_x exists in a nbd. of (a, b), therefore on taking the limit as $h > 0$, we get

$$\frac{1}{k}[f_x(a, b + k) - f_x(a, b)] = \underset{h \to 0}{\text{Lt}} f_{xy}(a + \theta' h, b + \theta k).$$

Now, let $k \to 0$.

Then $$\underset{k \to 0}{\text{Lt}} \frac{f_x(a, b + k) - f_x(a, b)}{k} = \underset{k \to 0}{\text{Lt}} \; \underset{h \to 0}{\text{Lt}} f_{xy}(a + \theta' h, b + \theta k)$$

$\Rightarrow \quad f_{yx}(a, b) = f_{xy}(a, b)$; since f_{xy} is continuous at (a, b).

This proves the theorem.