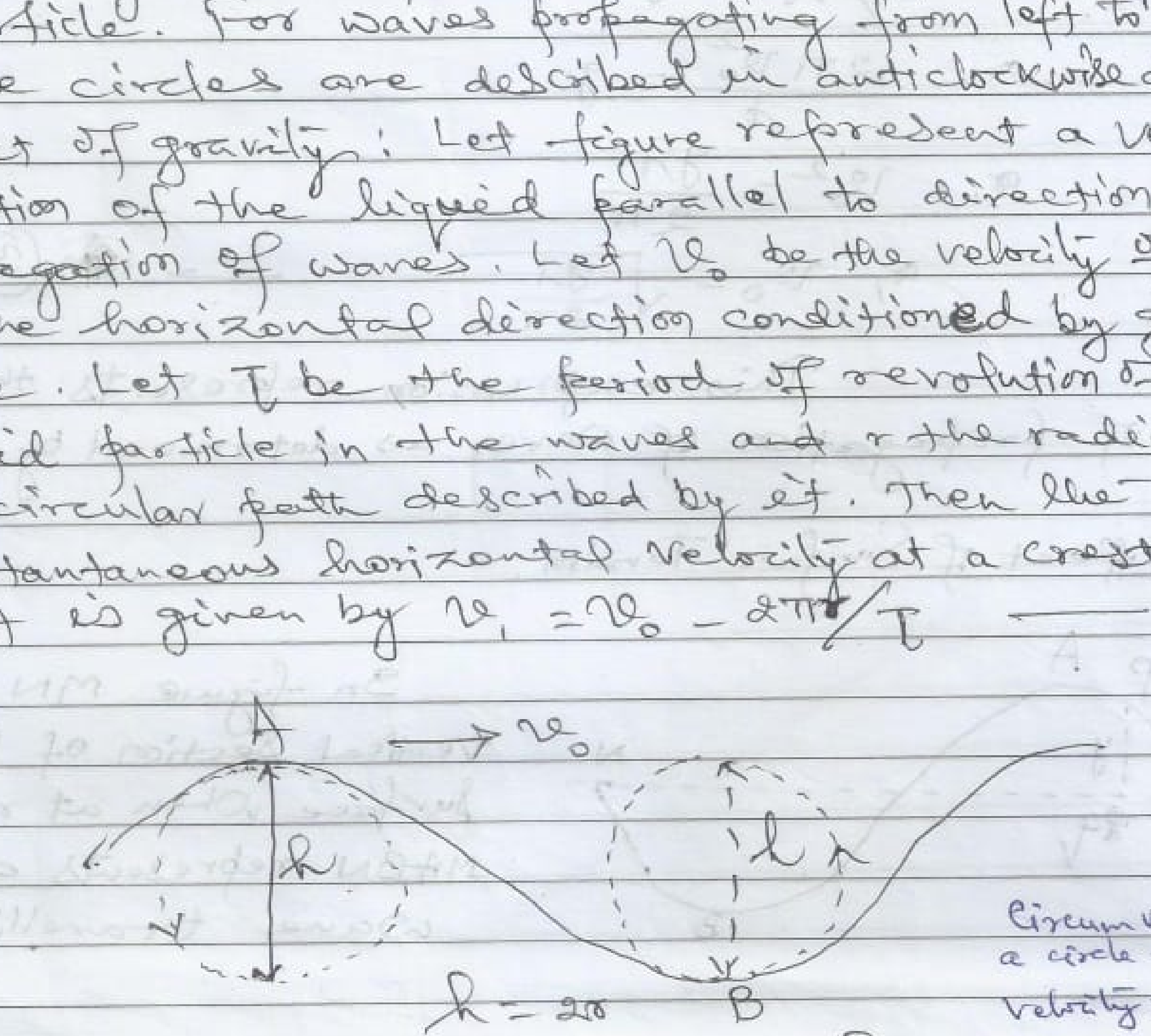
Velocity of propagation of waves on the Surface of liquid : Waves and Ripples

The velocity of waves formed on the Surface of a liquid is governed by the force of gravity as well as by the Surface tension of the liquid. If the wavelength of the waves is larger than a critical value, which is characteristic of the liquid, then their velocity is mainly governed by gravity. Such gravity waves are simply called 'waves'. If, however, the wavelength is smaller than the critical value, then the Surface tension becomes more important in deciding the velocity. The waves are then called as ripples.

If the amplitude of the waves is small, each particle of the liquid in the waves describes a circular path in the vertical plane, the waves propagate a distance equal to wavelength λ during one complete revolution of the liquid particle. For waves propagating from left to right the circles are described in anticlockwise direction.

1. Effect of gravity: Let figure represent a vertical section of the liquid parallel to direction of propagation of waves. Let v_0 be the velocity of waves in the horizontal direction conditioned by gravity alone. Let T be the period of revolution of a liquid particle in the waves and r the radius of circular path described by it. Then the instantaneous horizontal velocity at a crest A is given by $v_1 = v_0 - 2\pi r/T$ ——— (1)



$h = 2r$

Fig. 1

Circum of a circle = $2\pi r$
velocity = $\frac{dist.}{t}$
$= 2\pi$

or, $v_1^2 = v_0^2 + \frac{4\pi^2 r^2}{T^2} - \frac{4\pi v_0 r}{T}$

At the trough B, similarly, the velocity of the particle is given by

$$v_2 = v_0 + \frac{2\pi}{T} r,$$

$$\text{or, } v_2^2 = v_0^2 + \frac{4\pi^2 r^2}{T^2} + \frac{4\pi v_0 r}{T}$$

$$\therefore\ v_2^2 - v_1^2 = \frac{8\pi v_0 r}{T} \quad \text{—— (1)}$$

Assuming that the relative increase of velocity of particle v_2 over v_1 is due to its fall under the action of gravity through a height $h = 2r$, we get $v_2^2 = v_1^2 + 2g \cdot 2r$ $(\because h = 2r)$

$$\text{or, } v_2^2 - v_1^2 = 4gr \quad \text{—— (2)}$$

$\therefore$ From (1) and (2) $\frac{{}^{2}\cancel{8}\pi v_0 \cancel{r}}{T} = \cancel{4}g\cancel{r}$

$$\text{or, } \frac{2\pi v_0}{\lambda / v_0} = g \qquad (\because T = \lambda / v_0)$$

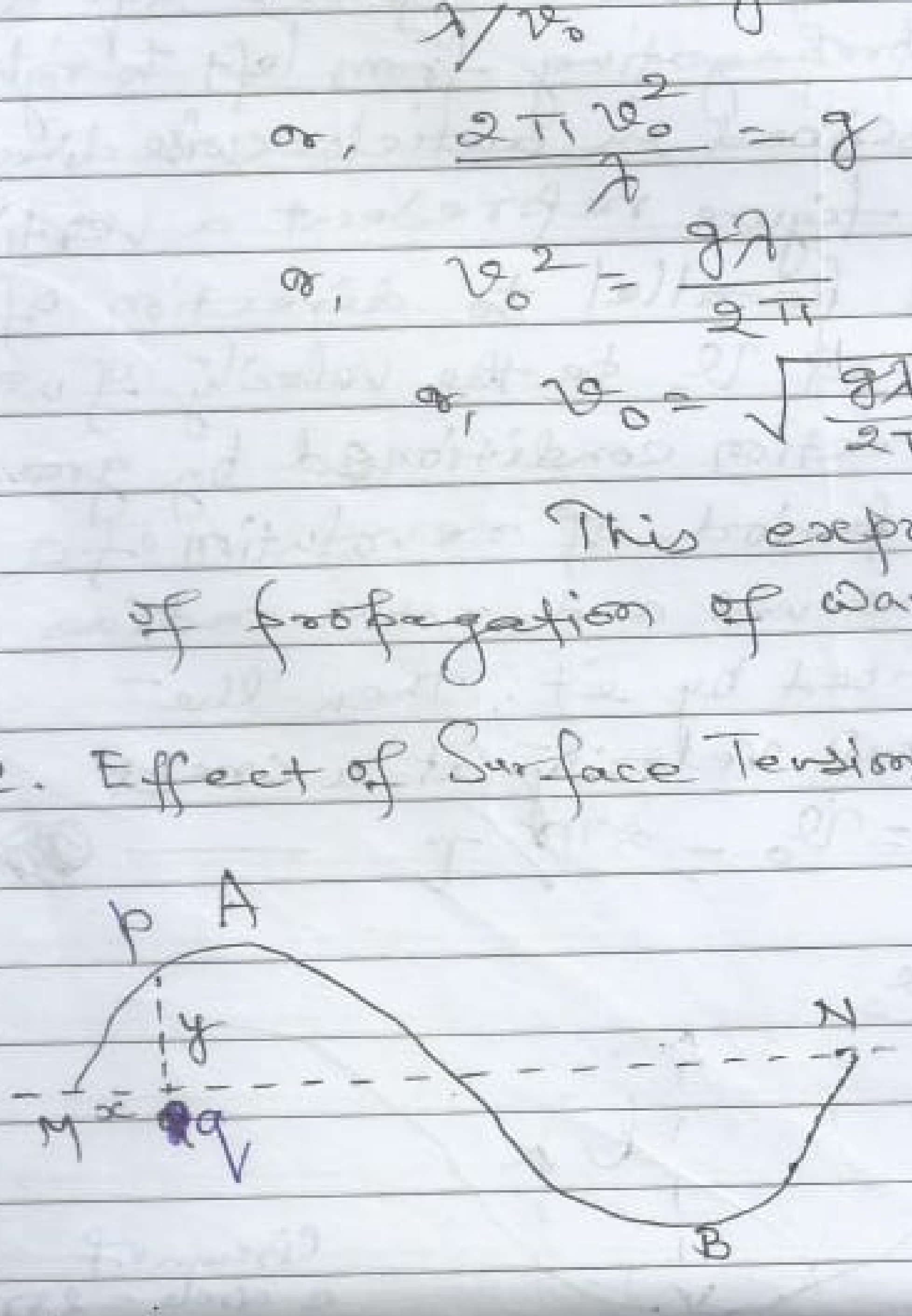
$$\text{or, } \frac{2\pi v_0^2}{\lambda} = g$$

$$\text{or, } v_0^2 = \frac{g\lambda}{2\pi}$$

$$\text{or, } v_0 = \sqrt{\frac{g\lambda}{2\pi}} \quad \text{—— (3)}$$

This expression represents the velocity of propagation of waves as determined by gravity alone.

2. Effect of Surface Tension:



In figure MN is a vertical section of the liquid surface when at rest and MABN represents a harmonic wave travelling

$T = \lambda / V$ $\quad \omega = 2\pi\nu = \frac{2\pi}{T} = \frac{2\pi V}{\lambda}$, $\therefore t = \frac{x}{V}$

$\therefore \omega t = \frac{2\pi V}{\lambda} \cdot \frac{x}{V} = \frac{2\pi}{\lambda}$

over the surface. Let y be the displacement of a particle originally at q at a distance x from some arbitrary origin M at any time t, then

$$y = a \sin(\omega t + \Phi) = a \sin\left(\frac{2\pi x}{\lambda} + \Phi\right) \quad - (4)$$

where a is the amplitude and Φ is the initial phase.

The vertical pressure at q due to gravity alone is increased by $y \rho g$ (ρ being density of liquid) and excess pressure at p on the concave side due to surface tension is T/R directed normally outwards (cylindrical film), where T is the surface tension of the liquid and R is the radius of curvature of the surface at p. For waves of amplitude less than the wavelength, the normal pressure at p is vertical, so that the net increase in vertical pressure at q when the waves pass over the liquid surface is given by

$$\text{Excess pressure} = y\rho g - \frac{T}{R} \quad - (5)$$

Differentiating eq^n (4), we get

$$\frac{d^2 y}{dx^2} = \frac{-4\pi^2 y}{\lambda^2}$$

But $\frac{1}{R} = \frac{d^2y/dx^2}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}} = \frac{d^2 y}{dx^2}$, for $y \ll 1$

$$\therefore \frac{1}{R} = -\frac{4\pi^2 y}{\lambda^2} \quad - (6)$$

Substituting this in eq^n (5), we get

$$\text{Excess pressure} = y\rho g + \frac{4\pi^2 y T}{\lambda^2}$$

$$= y\rho\left(g + \frac{4\pi^2 T}{\lambda^2 \rho}\right) \quad - (7)$$

But as already pointed out that the excess pressure due to gravity alone is $y\rho g$, therefore eq^n (7) indicates that the effect of surface tension is to increase the effective value of acceleration due to gravity from g to $(g + 4\pi^2 T/\lambda^2 \rho)$.

Hence the velocity of propagation of waves under the combined effect of gravity and surface tension, then using $V_0 = \sqrt{g\lambda/2\pi}$, we get

$$V = \sqrt{\frac{\lambda}{2\pi}\left(g + \frac{4\pi^2 T}{\lambda^2 \rho}\right)} = \sqrt{\frac{\lambda g}{2\pi} + \frac{2\pi T}{\lambda \rho}} \quad (8)$$

This expression shows that when $\lambda = 0$ or $\lambda = \infty$, $V = \infty$. In between these two extreme values of λ, there must be a certain value of it for which V has the minimum value. Clearly, the product of the two terms $\lambda g/2\pi$ and $2\pi T/\lambda.\rho$ viz gT/ρ is a constant. Therefore, by a theorem in algebra V will be minimum when the two terms are equal i.e. when

$$\frac{\lambda g}{2\pi} = \frac{2\pi T}{\lambda \rho} \quad \text{or, when } \lambda^2 . g\rho = 4\pi^2 T \quad (9)$$

$$\text{or, when } \lambda^2 = \frac{4\pi^2 T}{g\rho} \quad \text{or, when } \lambda = 2\pi\sqrt{\frac{T}{g\rho}}$$

This value of λ, for which the velocity of the wave is the minimum, is called the critical wave-length and may be denoted by the symbol λ_c.

Thus $\lambda_c = 2\pi\sqrt{T/g.\rho}$

Putting value of $T/\rho = \lambda^2 g/4\pi^2$ from (9) in (8)

$$V_m = \sqrt{\frac{\lambda g}{2\pi} + \frac{2\pi}{\lambda}.\frac{\lambda^2 g}{4\pi^2}} = \sqrt{\frac{\lambda g}{2\pi} + \frac{\lambda g}{2\pi}} = \sqrt{\frac{2\lambda g}{2\pi}} = \sqrt{\frac{\lambda g}{\pi}}$$

Now Substituting $2\pi\sqrt{T/g\rho}$ for λ, we have

$$V_m = \sqrt{\frac{2\pi\sqrt{\frac{T}{g\rho}}\cdot g}{\pi}} = \sqrt{2}\left(\frac{Tg^2}{g\rho}\right)^{1/4} = \sqrt{2}\left(\frac{T\cdot g}{\rho}\right)^{1/4}$$

$$\therefore V_m = \sqrt{2}\left(T\cdot g/\rho\right)^{1/4}$$

from eq^n (8)

(i) If $\lambda > \lambda_c$, the first term $\lambda g/2\pi$ becomes more important and λ increases and therefore neglecting the second term in comparison with it, we have

$$V = \sqrt{\lambda g/2\pi}$$

Disturbance of this type, whose wave-length is greater than the critical value are known as gravity waves, their propagation being mainly due to the force of gravity and their velocity increases as λ increases.

(ii) If $\lambda < \lambda_c$ the Second term becomes more predominant and the first term may be neglected. So that in this case $V = \sqrt{2\pi T/\lambda\rho}$

Waves of this type for which the wave length is less than the critical wave-length, are called ripples or capillarity waves. Their propagation is, in main due to Surface tension and their velocity decreases as λ increases.